AXIALLY SYMMETRIC STATIONARY THERMAL INTERACTION OF TWO BOUNDED CYLINDERS

G. B. Petrazhitskii and A. M. Pylaev

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The analytical solution of a two-dimensional problem is discussed for general linear boundary conditions with an arbitrary distribution of heat sources both inside the cylinders and on their plane of contact. The main difficulty in applying the method of separation of variables in the present case is connected with taking into account the "contact" conditions between the two cylinders.

The present problem is of interest when calculations are to be made of the temperature fields in radioelectronic instruments. The results we obtain can also be used to analyze thermal processes in nuclear power engineering and in solving various technological problems (welding by friction, et al [1]).

We introduce the following variables and defining parameters of the process in question:

$$\Theta_i = \frac{t_i}{t_c},\quad \zeta_i = \frac{|z_i|}{R},\quad \rho = \frac{r}{R},\quad h_i = \frac{\beta_i R}{\lambda_i},\quad g_i = \frac{\gamma_i R}{\lambda_i},\quad g = \frac{\gamma R}{\lambda_1},$$
$$u_i = \frac{W_i R^2}{t_c \lambda_i},\quad u = \frac{WR}{t_c\lambda_1},\quad \delta_i = \frac{d_i}{R},\quad \lambda = \frac{\lambda_2}{\lambda_1},\quad i = 1,\ 2. \tag{1}$$

We assume that the volume (u_i) and surface (u) dimensionless heat source intensities depend on (ζ_i and ρ) and on ρ, respectively. We consider the boundary problem:

$$\frac{\partial^2\Theta_i}{\partial\rho^2} + \frac{\partial\Theta_i}{\rho\,\partial\rho} + \frac{\partial^2\Theta_i}{\partial\zeta_i^2} = u_i \qquad (\zeta_i \in [0,\ \delta_i],\quad \rho \in [0,\ 1]), \tag{2}$$

$$\left.\frac{\partial\Theta_i}{\partial\rho}\right|_{\rho=1} + h_i\Theta_i|_{\rho=1} = h_i,\quad \Theta_i|_{\rho=0} < \infty, \tag{3}$$

$$\left.\frac{\partial\Theta_i}{\partial\zeta_i}\right|_{\zeta_i=\delta_i} + g_i\Theta_i|_{\zeta_i=\delta_i} = g_i, \tag{4}$$

$$\Theta_2|_{\zeta_2=0} - \Theta_1|_{\zeta_1=0} = \frac{1}{g}\cdot\left.\frac{\partial\Theta_2}{\partial\zeta_2}\right|_{\zeta_2=0}, \tag{5}$$

$$-\left.\frac{\partial\Theta_1}{\partial\zeta_1}\right|_{\zeta_1=0} = \lambda\left.\frac{\partial\Theta_2}{\partial\zeta_2}\right|_{\zeta_2=0} + u. \tag{6}$$

Anticipating the possibility of using the method of separation of variables, we seek a solution of the system (2)-(6) in the form

$$\Theta_i = 1 + \sum_{n=1}^{\infty} R_{in}(\rho) Z_{in}(\zeta_i), \tag{7}$$

where $R_{in}(\rho)$ and $Z_{in}(\zeta_i)$ satisfy, respectively, the homogeneous boundary conditions obtainable from Eqs. (3) and (4).

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For the $R_{in}(\rho)$ we can take the Bessel functions [2] of zero order of the first kind, namely $J_0(\alpha_{in}\rho)$ where the α_{in} are the roots of the equation

$$\alpha_{in}J_1(\alpha_{in}) - h_i J_0(\alpha_{in}) = 0. \tag{8}$$

Assuming that u_i for $0 \leq \rho \leq 1$ has the expansion

$$u_i = \sum_{n=1}^{\infty} J_0(\alpha_{in}\rho)\, u_{in}, \tag{9}$$

we obtain from Eq. (2) the following set of differential equations for the $Z_{in}(\zeta_i)$:

$$\frac{d^2 Z_{in}}{d\zeta_i^2} - \alpha_{in}^2 Z_{in} = u_{in}, \quad i = 1,\ 2;\ \ n = 1 - \infty. \tag{10}$$

Let $\varphi_{in}(\zeta_i)$ be an arbitrary particular solution of Eq. (10). Then the general solution of Eq. (10) which takes into account the boundary condition (4) can be written in the form

$$Z_{in} = C_{in}\left[\operatorname{ch}\alpha_{in}(\zeta_i - \delta_i) - \frac{g_i}{\alpha_{in}}\operatorname{sh}\alpha_{in}(\zeta_i - \delta_i)\right] + \Phi_{in}(\zeta_i), \tag{11}$$

where

$$\Phi_{in}(\zeta_i) = \varphi_{in}(\zeta_i) - \frac{g_i\varphi_{in}(\delta_i) + \varphi'_{in}(\delta_i)}{\alpha_{in} + g_i}\exp\left[(\zeta_i - \delta_i)\alpha_{in}\right], \tag{12}$$

wherein

$$\varphi'_{in}(\delta_i) = \left.\frac{d\varphi_{in}(\zeta_i)}{d\zeta_i}\right|_{\zeta_i = \delta_i}$$

The main difficulty in solving the problem is that of determining the constants C_{in} in taking into account the conditions of "contact."

We use the representation of the function $J_0(\alpha_{in}\rho)$ in the form of an infinite series in terms of the functions $J_0(\alpha_{jm}\rho)$ for $0 < \rho < 1$, $i \neq j$ and for $m = 1, 2, \ldots$ Let

$$J_0(\alpha_{1n}\rho) = \sum_{m=1}^{\infty} b_{nm}J_0(\alpha_{2m}\rho) \quad \text{and} \quad J_0(\alpha_{2n}\rho) = \sum_{m=1}^{\infty} d_{nm}J_0(\alpha_{1m}\rho). \tag{13}$$

Taking Eq. (8) into account, we obtain, using formulas of the type [3]

$$b_{nm} = \int_0^1 \rho J_0(\alpha_{2m}\rho)\, J_0(\alpha_{1n}\rho)\, d\rho \left[\int_0^1 \rho J_0^2(\alpha_{1m}\rho)\right]^{-1}, \tag{14}$$

the results

$$b_{nm} = \frac{2(h_2 - h_1)\, J_0(\alpha_{1n})\, \alpha_{2m}^2}{(\alpha_{2m}^2 - \alpha_{1n}^2)\, J_0(\alpha_{2m})\, (\alpha_{2m}^2 + h_2^2)}$$

and

$$d_{nm} = \frac{2(h_2 - h_1)\, \alpha_{1m}^2 J_0(\alpha_{2n})}{(\alpha_{2n}^2 - \alpha_{1m}^2)\, (\alpha_{1m}^2 + h_1^2)\, J_0(\alpha_{1m})}.$$

Let us assume also that

$$u = \sum_{n=1}^{\infty} J_0(\alpha_{1n}\rho)\, u_n. \tag{15}$$

From Eq. (6) we obtain, taking note of Eqs. (13) and (15),

$$C_{1n} = Q_{1n}^{-1}\left[\psi_n - \lambda\sum_{m=1}^{\infty} C_{2m} d_{mn} Q_{2m}\right], \tag{16}$$

where

$$\psi_n = u_n + \Phi_{1n}' + \lambda \sum_{m=1}^{\infty} d_{mn}\Phi_{2m}',$$
$$\Phi_{im}' = \left.\frac{d\Phi_{im}(\zeta_i)}{d\zeta_i}\right|_{\zeta_i=0} \tag{17}$$

and

$$Q_{in} = \alpha_{in}\,\mathrm{sh}\,\alpha_{in}\delta_i + g_i\,\mathrm{ch}\,\alpha_{in}\delta_i. \tag{18}$$

Finally, using Eqs. (13) and (16), we deduce from Eq. (5) the C_{2n};

$$C_{2n} = P_{2n}^{-1}\left[\Psi_n - \lambda \sum_{l=1}^{\infty} C_{2l}Q_{2l} \sum_{m=1}^{\infty} b_{mn}\, d_{lm}\, P_{1m}\, Q_{1m}^{-1}\right], \tag{19}$$

where

$$\Psi_n = -\Phi_{2n}(0) + \sum_{m=1}^{\infty} b_{mn}\,[\Phi_{1m}(0) + P_{1m}\psi_m Q_{1m}^{-1}] + \Phi_{2n}' g^{-1}$$

and

$$P_{im} = \mathrm{ch}\,\alpha_{im}\delta_i + \frac{g_i}{\alpha_{im}}\,\mathrm{sh}\,\alpha_{im}\delta_i + \left(\frac{Q_{2m}}{g}\right)^{i-1} - (2-i). \tag{20}$$

Introducing the notation

$$B_n = C_{2n}Q_{2n}J_0(\alpha_{2n}),\quad T_{m,ln} = \frac{P_{1m}(\alpha_{2l}^2 - \alpha_{1m}^2)^{-1}\alpha_{1m}^2}{Q_{1m}(\alpha_{1m}^2 + h_1^2)(\alpha_{2n}^2 - \alpha_{1m}^2)},$$
$$F_n = \sum_{m=1}^{\infty} T_{m,nn} + \frac{(\alpha_{2n}^2 + h_2^2)P_{2n}}{4\lambda(h_2 - h_1)^2\alpha_{2n}^2 Q_{2n}},\quad V_n = \frac{J_0(\alpha_{2n})\Psi_n(\alpha_{2n}^2 + h_2^2)}{4\lambda(h_2 - h_1)^2\alpha_{2n}^2 F_n},$$

we obtain an infinite system of equations, linear in the B_n,

$$B_n + \sum_{l=1(n)}^{\infty} B_l \sum_{m=1}^{\infty} T_{m,ln} F_n^{-1} = V_n, \tag{19a}$$

where $\sum\limits_{l=1(n)}^{\infty} a_l = \sum\limits_{l=1}^{\infty} a_l - a_n$. We show that

$$\sigma = \sum_{n=1}^{\infty}\sigma_n = \sum_{n=1}^{\infty}\sum_{l=1(n)}^{\infty}\left(\sum_{m=1}^{\infty}\frac{T_{m,ln}}{F_n}\right)^2 < \infty. \tag{21}$$

We note from Eq. (8) and from Eqs. (18) and (20) that

$$\alpha_{i(n+1)} \approx \pi + \alpha_{in} \quad\text{and}\quad \frac{\mathrm{th}\,\alpha_{2n}\delta_2}{\alpha_{2n}} < \frac{P_{2n}}{Q_{2n}};\quad \frac{P_{1n}}{Q_{1n}} < \frac{1}{\alpha_{1n}\,\mathrm{th}\,\alpha_{1n}\delta_1}.$$

Therefore

$$F_n^2 > \frac{K_1}{\alpha_{2n}^2}, \tag{22}$$

where $K_1 = \min\limits_n [(\mathrm{th}\,\alpha_{2n}\delta_2 - \alpha_{2n}g^{-1})/(4\lambda(h_2-h_1)^2)]^2$. Along with this we have

$$\left|\sum_{m=1}^{\infty} T_{m,ln}\right| < |T_{n,ln} - T_{l,ln}| + S_{ln};\quad S_{ln} = S_1^{(\beta-1)} + S_{(\beta+1)}^{(\gamma-1)} + S_{(\gamma+1)}^{\infty},$$

where

$$\beta = \min(l,\ n), \quad \gamma = \max(l,\ n)$$

and

$$S_a^b = \frac{1}{2\pi\ \mathrm{th}\,\alpha_{11}\delta_1} \sum_{m=a}^{b} \frac{\alpha_{1m}(\alpha_{1m}^2 + h_1^2)^{-1}}{|\alpha_{2n}^2 - \alpha_{1m}^2|\ |\alpha_{2l}^2 - \alpha_{1m}^2|}$$

$$\approx \frac{1}{2\pi\ \mathrm{th}\,\alpha_{11}\delta_1} \int\limits_{\left(\alpha_{1a} - \frac{\pi}{2}\right)^2}^{\left(\alpha_{1b} + \frac{\pi}{2}\right)^2} \frac{dx\,(x + h_1^2)^{-1}}{|\alpha_{2n}^2 - x|\ |\alpha_{2l}^2 - x|}.$$

Using the last relation we find that

$$S_{ln} < \frac{K_2}{(\alpha_{2n}^2 + h_1^2)(\alpha_{2l}^2 + h_1^2)} + \frac{K_3}{|\alpha_{2n}^2 - \alpha_{1l}^2|} \left| \frac{K_4 + \ln\alpha_{2n}}{\alpha_{2n}^2 + h_1^2} + \frac{K_4 + \ln\alpha_{2l}}{\alpha_{2l}^2 + h_1^2} \right|, \tag{23}$$

where the K_j are bounded and independent of l and n. In addition,

$$|T_{n,ln} - T_{l,ln}| < \frac{K_5}{\alpha_{1n}\alpha_{1l}|\alpha_{2n} - \alpha_{1l}|},$$

where

$$K_5 \leqslant (\mathrm{cth}\,\alpha_{11}\delta_1) \max_m (\alpha_{2m}^2 - \alpha_{1m}^2)^{-2}, \quad m = 1 - \infty. \tag{24}$$

Here we can show boundedness of K_5 by use of the relation

$$(\alpha_{2m} - \alpha_{1m}) + \mathrm{tg}(\alpha_{2m} - \alpha_{1m}) \left[h_1 + 1 + \frac{\alpha_{2m}^2}{h_2 + 1} \right] \approx \alpha_{2m} \left(\frac{h_2 - h_1}{h_2 + 1} \right),$$

obtainable from Eq. (8) for large m, since

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} \right).$$

Using Eqs. (22)-(24) and also the obvious inequality

$$\left(\sum_{l=1}^{L} a_l\right)^2 < L \sum_{l=1}^{L} a_l^2,$$

we deduce that

$$\frac{\sigma_n}{\alpha_{2n}^2} < \frac{M_2}{(\alpha_{2n}^2 + h_1^2)^2} \int\limits_{\left(\alpha_{21} - \frac{\pi}{2}\right)}^{\infty} \frac{dx}{(x^2 + h_1^2)^2}$$

$$+ M_3 M\,[(\alpha_{2n}^2 + h_1^2)^{-2} S_{(2,0)} + S_{(2,2)}] + M_5 S_{(1,2)},$$

where

$$M_j = \frac{4K_j}{\pi K_1}, \quad M = \max_m \left[\frac{(K_4 + \ln\alpha_{2m})^2}{\alpha_{2m}^2 + h_1^2} \right]$$

and

$$S_{(a,b)} \approx \int\limits_{\left(\alpha_{21} - \frac{\pi}{2}\right)}^{\left(\alpha_{2n} - \frac{\pi}{2}\right)} \frac{dx}{(\alpha_{2n}^a - x^a)^2 x^b} + \int\limits_{\left(\alpha_{2n} + \frac{\pi}{2}\right)}^{\infty} \frac{dx}{(\alpha_{2n}^a - x^a)^2 x^b}.$$

It may be shown that $\sigma_n < K/\alpha_{2n}^2$ and the satisfaction of inequality (21) then becomes obvious.

It is known [6] that a system of the type (19a), subject to the condition (21), has a unique determinate solution for an arbitrary set of V_n if

$$0 < \sum_{n=1}^{\infty} V_n^2 < \infty; \quad \text{wherein} \quad \sum_{n=1}^{\infty} B_n^2 < \infty.$$

In finding the solution of the system (19a) it is convenient to combine iteration of the values of B_n for $n > N$, if $\sum_{n=N}^{\infty} \sigma_n < 1$, with the use of direct methods (for example, the method of Gaussian elimination) for the subsystem of the system (19a), consisting of the first N equations.

Finding C_{2n}, subsequent to determining C_{1n} from Eq. (16) and substitution of C_{in} into Eq. (11), we finally obtain an analytic expression for Θ_i. The temperature at an arbitrary point of the space ($z \in [-d_1, d_2]$, $r \in [0, R]$) can be calculated from Eqs. (7) and (11), with Eq. (1) taken into account.

The solution of the problem is substantially simplified when

$$h_1 \ll 1 \quad \text{and} \quad g_1 \frac{d_1}{R} \ll 1, \tag{25}$$

conditions which are often realized when materials of sharply differing thermophysical properties are in contact (metals and heat insulators), and when there is a moderate intensity of heat transfer with the surrounding medium.

When the conditions (25) are satisfied, we usually have the valid expansions

$$\Theta_i \approx 1 + \sum_{n=1}^{N_i} Z_{in}(\zeta_i) J_0(\alpha_{in}\rho) \quad \text{for} \quad N_1 = 1. \tag{26}$$

By Cramer's Rule we obtain

$$C_{2n} \approx P_{2n}^{-1} \left[\Psi_n - \frac{E_n}{DJ_0(\alpha_{2n})} \sum_{m=1}^{N_2} \frac{Q_{2m}\Psi_m J_0(\alpha_{2m})}{P_{2m}(\alpha_{2m}^2 - \alpha_{11}^2)} \right], \tag{27}$$

where

$$E_n = \frac{4\lambda (h_2 - h_1)^2 \alpha_{11}^2 \alpha_{2n}^2 P_{11}}{(\alpha_{11}^2 + h_1^2)(\alpha_{2n}^2 - \alpha_{11}^2)(\alpha_{2n}^2 + h_2^2) Q_{11}}, \quad D = 1 + \sum_{m=1}^{N_2} \frac{E_m Q_{2m}}{(\alpha_{2m}^2 - \alpha_{11}^2) P_{2m}}.$$

For the special case in which u_i and u have constant values, we can determine the $\varphi_{in}(\zeta_i)$ with the aid of the relations

$$1 = \sum_{n=1}^{\infty} \frac{2J_1(\alpha_{in}) J_0(\alpha_{in}\rho)}{\alpha_{in}[J_0^2(\alpha_{in}) + J_1^2(\alpha_{in})]} = \sum_{n=1}^{\infty} \frac{2h_i J_0(\alpha_{in}\rho)}{(\alpha_{in}^2 + h_i^2) J_0(\alpha_{in})},$$

satisfied for $0 \leq \rho \leq 1$ and $h_i > 0$ [4] for the roots of Eq. (8). Then

$$\varphi_{in} = \frac{-2h_i u_i}{\alpha_{in}^2(\alpha_{in}^2 + h_i^2) J_0(\alpha_{in})}; \quad \Phi_{in} = \varphi_{in}\left(1 - \frac{g_i \exp[\alpha_{in}(\zeta_i - \delta_i)]}{\alpha_{in} + g_i}\right) \tag{28}$$

and

$$\Psi_n = \frac{2h_2}{\alpha_{2n}^2(\alpha_{2n}^2 + h_2^2) J_0(\alpha_{2n})} \left\{ u_2 \left(1 + \frac{4(h_2 - h_1)^2 \lambda P_{11} \alpha_{11}^2 g_2 \alpha_{2n}^4}{Q_{11}(\alpha_{2n}^2 - \alpha_{11}^2)(\alpha_{11}^2 + h_1^2)} \right.\right.$$

$$\times \sum_{m=1}^{N_2} [\alpha_{2m}(\alpha_{2m}^2 + h_2^2)(\alpha_{2m}^2 - \alpha_{11}^2)(\alpha_{2m} + g_2) \exp(\alpha_{2m}\delta_2)]^{-1}$$

$$- \frac{g_2 \exp(-\alpha_{2n}\delta_2)}{(\alpha_{2n}+g_2)(1-\alpha_{2n}g^{-1})^{-1}}\Bigg) + \frac{2u_1(h_2-h_1)h_1\alpha_{2n}^4}{h_2\alpha_{11}^2(\alpha_{11}^2+h_1^2)(\alpha_{2n}^2-\alpha_{11}^2)}$$

$$\times\left[1-\frac{g_1\exp(-\alpha_{11}\delta_1)}{\alpha_{11}+g_1}\left(1+\frac{\alpha_{11}P_{11}}{Q_{11}}\right)+\frac{uP_{11}}{u_1Q_{11}}\right]\Bigg\}. \quad (29)$$

Thus the temperature field of the physical system in which the heat source intensities (u_1, u_2, and u) are constant can, when the conditions (25) apply, be calculated from the formulas (26) when account is taken of the relations (11), (27)-(29), (18) and (20). The method we have described can, in conjunction with the Laplace transformation, also be used in solving corresponding nonstationary problems. However, in view of their complexity, the resulting formulas can be conveniently applied only in finding a first approximation to the solution of the problem, corresponding to the so-called regular state of the system.

NOTATION

t_i temperature of i-th cylinder (i = 1, 2);
t_c ambient temperature;
β_i, γ_i heat transfer coefficient to medium;
γ^{-1} contact thermal resistance between cylinders;
λ_i thermal conductivity;
R radius;
d_i height of cylinder.

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